LING4400: Lecture Notes 5
Generalized Quantifiers

Contents

5.1 Cardinality [Cantor, 1880]

We must ensure our logic is expressive enough to model **proportional** quantifiers: *half*, *most*, etc.

(1) *Most Brazilian states are tropical.*

These need Cardinality functions of type $\langle\langle e,t\rangle,e\rangle$ that count the **True** outputs in the input function:

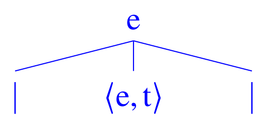
$$[\![\mathsf{Cardinality}]\!]^M = \left[\begin{array}{ccc} \text{input} & & \text{output} \\ \hline \left[\begin{array}{ccc} \text{input} & & \text{output} \\ \hline \textbf{Laos} & : & \textbf{False} \\ \textbf{Mali} & : & \textbf{False} \\ \textbf{Togo} & : & \textbf{False} \\ \vdots & \vdots & \vdots \end{array}\right] & : & \mathbf{0} \\ \left[\begin{array}{ccc} \text{input} & & \text{output} \\ \hline \textbf{Laos} & : & \textbf{True} \\ \textbf{Mali} & : & \textbf{False} \\ \textbf{Togo} & : & \textbf{False} \\ \vdots & \vdots & \vdots \end{array}\right] & : & \mathbf{1} \\ \vdots & : & \mathbf{1} \\ (\langle e,t\rangle \text{ functions with one } \textbf{True} \text{ output}) & : & \mathbf{1} \\ \vdots & : & \mathbf{1} \\ \vdots & : & \mathbf{2} \\ (\langle e,t\rangle \text{ functions with two } \textbf{True} \text{ outputs}) & : & \mathbf{2} \\ \vdots & : & \mathbf{2} \\ \vdots & : & \vdots \end{array}\right].$$

(The output numbers **1**, **2**, etc., are entities that must belong to the domain of entities D_e^M.)

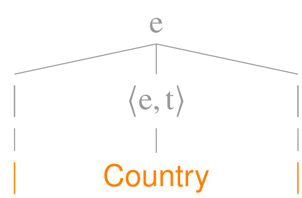
Cardinality is also notated as a **circumfix** operator – vertical bars around the argument:

$$[\![|s|]\!]^M = [\![\mathsf{Cardinality}\ s]\!]^M.$$

We can draw trees for expressions in circumfix notation using flattened rules:



For example:



Practice 5.1: cardinality of functions

Given the following denotations:

$$[\![\mathsf{Coastal}]\!]^M = \{\mathbf{Togo}, \mathbf{Peru}, \mathbf{Pune}\}$$
$$[\![\mathsf{Country}]\!]^M = \{\mathbf{Togo}, \mathbf{Peru}, \mathbf{Mali}\},$$

what is the denotation of the following expression?

$$[\![\ |\ \lambda_x\ \mathsf{Coastal}\ x \vee \mathsf{Country}\ x\ |\ ]\!]^M$$

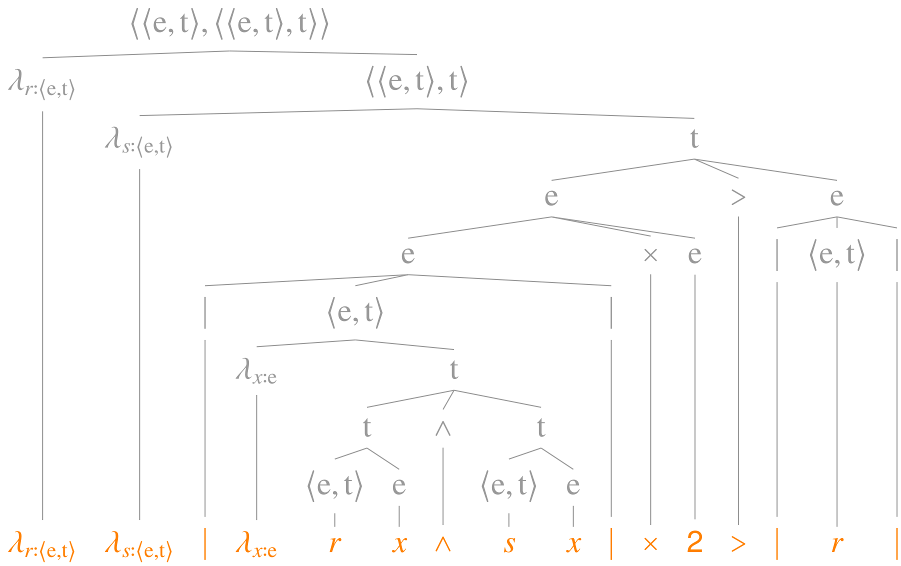
5.2 Generalized quantifier functions [Barwise & Cooper, 1981]

Generalized quantifiers of type $\langle\langle e, t\rangle, \langle\langle e, t\rangle, t\rangle\rangle$ compare cardinalities of $\langle e, t\rangle$ functions:

$$[\![\mathsf{Most}]\!]^M = [\![\ \underbrace{\lambda_{r:\langle e,t\rangle}}_{\text{restrictor}}\ \underbrace{\lambda_{s:\langle e,t\rangle}}_{\text{nuclear scope}}\ |\underbrace{\lambda_{x:e}\ r\ x \wedge s\ x}_{\text{intersection}}| \times 2 > |r|\ ]\!]^M.$$

Arguments r, s of a generalized quantifier are called the **restrictor** and **nuclear scope**, respectively.

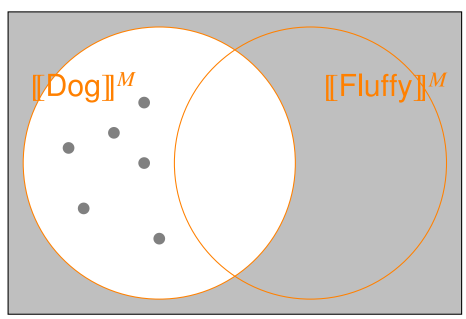
Here's how to make sense of that equation:



Generalized quantifiers compare denotations of intersections of restrictor r and nuclear scope s:

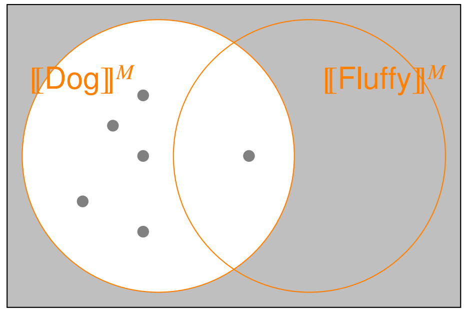
1. **Cardinal** quantifiers compare cardinalities of intersections $\lambda_{x:e}\ r\ x \wedge s\ x$ to absolute numbers:
   - $[\![\text{None}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| = 0]\!]^M$ — true if none of the r's are s's.

     For example, here's a Venn diagram for $\text{None}\ (\lambda_{x:e}\ \text{Dog}\ x)\ (\lambda_{x:e}\ \text{Fluffy}\ x)$:

     This expresses that *None of the dogs are fluffy*.
   - $[\![\text{ExactlyOne}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| = 1]\!]^M$ — true if one of the r's is an s.

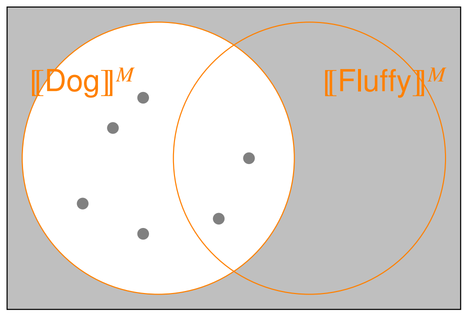
     For example, here's a Venn diagram for $\text{ExactlyOne}\ (\lambda_{x:e}\ \text{Dog}\ x)\ (\lambda_{x:e}\ \text{Fluffy}\ x)$:



This expresses that *Exactly one of the dogs are fluffy*.

- $[\![\mathsf{ExactlyTwo}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| = 2]\!]^M$ — true if two of the r's are s's.

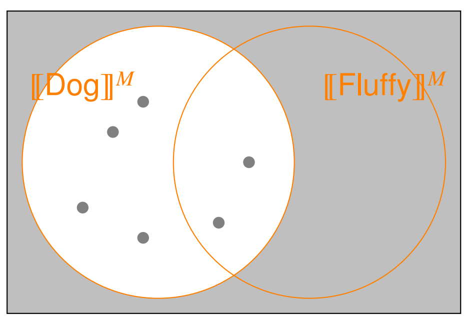
  For example, here's a Venn diagram for $\mathsf{ExactlyTwo}\ (\lambda_{x:e}\ \mathsf{Dog}\ x)\ (\lambda_{x:e}\ \mathsf{Fluffy}\ x)$:

  This expresses that *Exactly two of the dogs are fluffy*.

- $[\![\mathsf{Some}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| > 0]\!]^M$ — true if some of the r's are s's.

  For example, here's a Venn diagram for $\mathsf{Some}\ (\lambda_{x:e}\ \mathsf{Dog}\ x)\ (\lambda_{x:e}\ \mathsf{Fluffy}\ x)$:

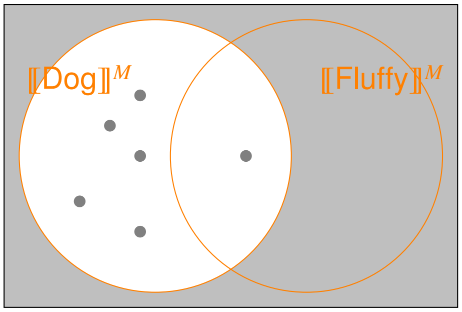
  This expresses that *Some of the dogs are fluffy*.

  (We'll see later this does the same thing as a first-order existential quantifier.)

2. **Proportional** quantifiers compare them to cardinalities of restrictors:

- $[\![\mathsf{Few}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| \times 2 < |r|]\!]^M$ — true if fewer than half the r's are s's.

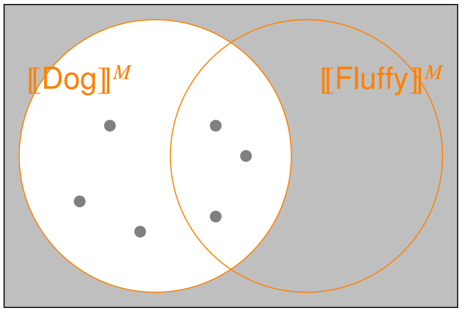
  For example, here's a Venn diagram for $\mathsf{Few}\ (\lambda_{x:e}\ \mathsf{Dog}\ x)\ (\lambda_{x:e}\ \mathsf{Fluffy}\ x)$:



This expresses that *Few of the dogs are fluffy*.

- $[\![\mathsf{Half}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| \times 2 = |r|]\!]^M$ — true if half of the r's are s's.

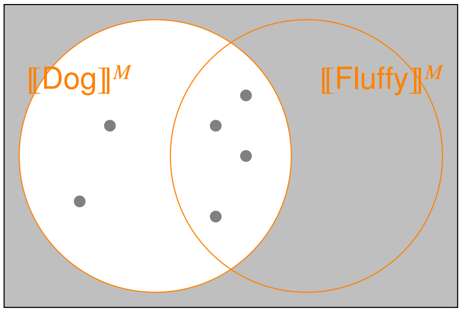
  For example, here's a Venn diagram for $\mathsf{Half}\ (\lambda_{x:e}\ \mathsf{Dog}\ x)\ (\lambda_{x:e}\ \mathsf{Fluffy}\ x)$:



  This expresses that *Half of the dogs are fluffy*.

- $[\![\mathsf{Most}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| \times 2 > |r|]\!]^M$ — true if more than half the r's are s's.

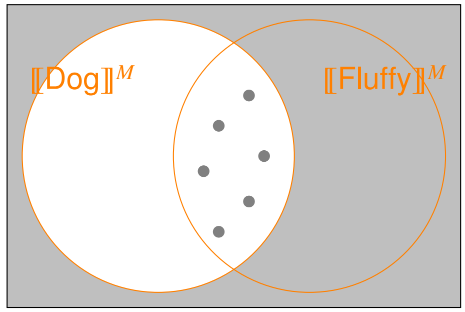
  For example, here's a Venn diagram for $\mathsf{Most}\ (\lambda_{x:e}\ \mathsf{Dog}\ x)\ (\lambda_{x:e}\ \mathsf{Fluffy}\ x)$



  This expresses that *Most of the dogs are fluffy*.

- $[\![\mathsf{All}]\!]^M = [\![\lambda_{r:\langle e,t\rangle}\ \lambda_{s:\langle e,t\rangle}\ |\lambda_{x:e}\ r\ x \wedge s\ x| \times 1 = |r|]\!]^M$ — true if all of the r's are s's.

For example, here's a Venn diagram for All $(\lambda_{x:\mathrm{e}}$ Dog $x)$ $(\lambda_{x:\mathrm{e}}$ Fluffy $x)$:

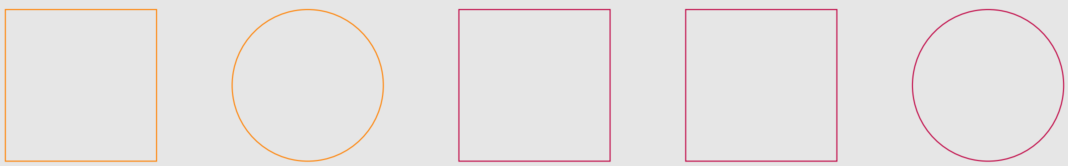


This expresses that *All of the dogs are fluffy*.

(We'll see later this does the same thing as a first-order universal quantifier.)

Practice 5.2: meaning

Given a world M of Shape entities (where Purple and Square have their usual meanings):



what is the denotation of the following lambda calculus expression?

$$[\![\ \text{Most}\ (\lambda_x\ \text{Shape}\ x\ \wedge \text{Purple}\ x)\ (\lambda_x\ \text{Square}\ x)\ ]\!]^M$$

Practice 5.3: another meaning

Given the same world of shapes above, what is the denotation of the following lambda calculus expression?

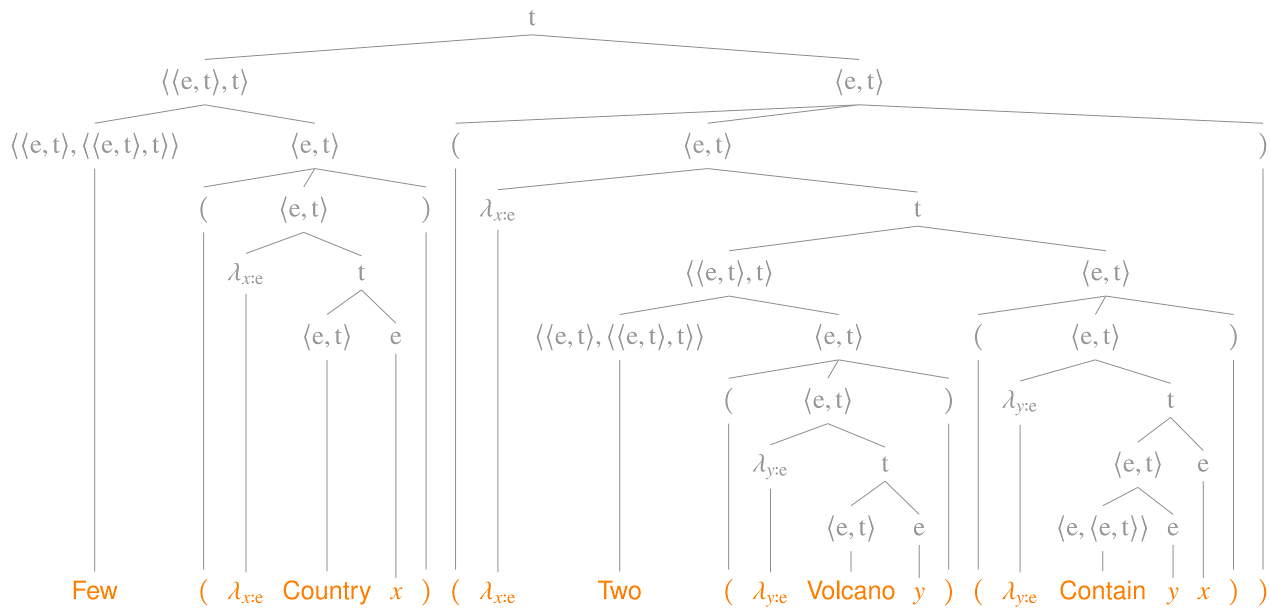
$$[\![\ \text{Most}\ (\lambda_x\ \text{Shape}\ x)\ (\lambda_x\ \text{Square}\ x\ \wedge \text{Purple}\ x)\ ]\!]^M$$

5.3 Building complex expressions out of generalized quantifiers

We can understand how logical expressions are constructed by drawing them as **derivation trees**:

1. The **branches** are function abstractions and applications and are labeled with types:
2. The **leaves** are constants, variables and logical symbols. (Later, we will use words.)

For example, here's a logical representation of '*Few countries contain two volcanoes.*':



Practice 5.4: tree drawing

Draw a derivation tree for the following expression:

$$\textsf{Most}\ (\lambda_x\ \textsf{Shape}\ x)\ (\lambda_x\ \textsf{Square}\ x \wedge \textsf{Purple}\ x)$$

5.4 Conservativity

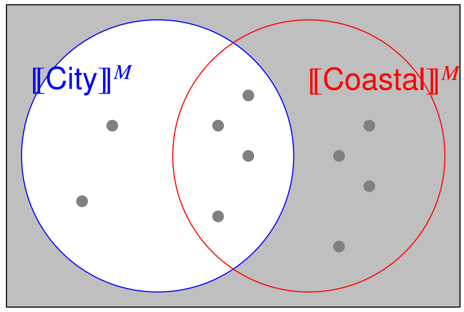
The following entailment holds of all generalized quantifiers:

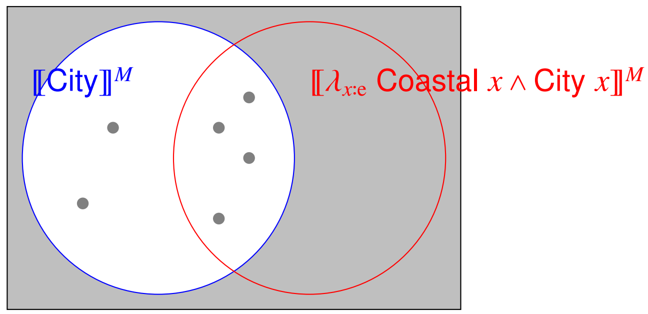
(2) a. *Most cities are coastal.*
  b. (entailed by 2a:) *Most cities are coastal cities.*

We can explain why with our analysis of generalized quantifiers:

$$\textsf{Most}\ (\lambda_x\ \textsf{City}\ x)\ (\lambda_x\ \textsf{Coastal}\ x)$$
$$\textsf{Most}\ (\lambda_x\ \textsf{City}\ x)\ (\lambda_x\ \textsf{Coastal}\ x \wedge \textsf{City}\ x)$$

These have the following Venn diagrams:





The extra constraint of City in red just removes entities from the non-blue part of the red circle.

This property is called **conservativity**.

5.5 Relation to probability

Proportional quantifiers represent can represent arbitrary conditional probabilities $\mathsf{P}_M(s|r)$.

This means we can use them for probabilistic reasoning.

For example, the following means *Most people who eat some mold are sick.*:

$$\mathsf{Most}\ (\lambda_{x:\mathrm{e}}\ \mathsf{Person}\ x \wedge \mathsf{Some}\ (\lambda_{y:\mathrm{e}}\ \mathsf{Mold}\ y)\ (\lambda_{y:\mathrm{e}}\ \mathsf{Eat}\ y\ x))\ (\lambda_{x:\mathrm{e}}\ \mathsf{Sick}\ x)$$

This means the same thing as the below probability expression (if you know probability notation):

$$\mathsf{P}_M(\lambda_{x:\mathrm{e}}\ \mathsf{Sick}\ x \mid \lambda_{x:\mathrm{e}}\ \mathsf{Person}\ x \wedge \mathsf{Some}\ (\lambda_{y:\mathrm{e}}\ \mathsf{Mold}\ y)\ (\lambda_{y:\mathrm{e}}\ \mathsf{Eat}\ y\ x)) > 0.5$$

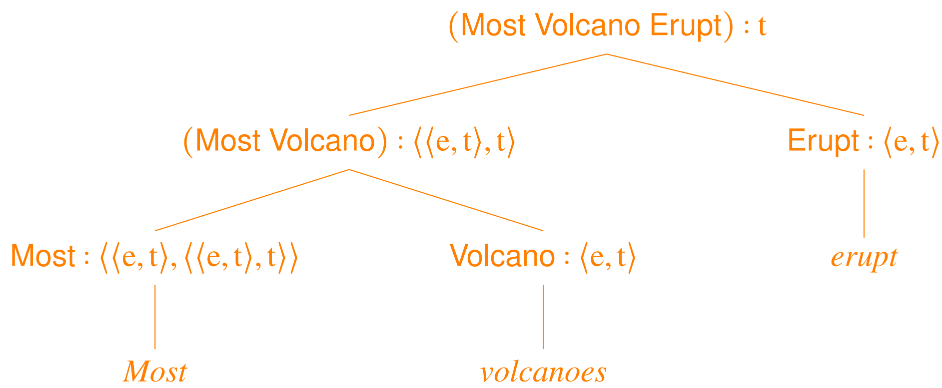
(The meaning is: *The probability that one is sick given that one eats mold is greater than 50%.*)

Probability statements like this can be used to reason and plan based on uncertain knowledge.

5.6 Composition of quantified noun phrases reverses function and argument

We can now associate words with the constant symbols at the leaves (bottoms) of our trees.

Predicate verb phrases are now arguments of subject noun phrases:



5.7 Generalized generalized quantifiers

We saw how to generalize many common quantifiers by whether they are cardinal or proportional.

We can further generalize many common quantifiers by their relation to their threshold number:

$$\begin{aligned}
[\![\textsf{Count}_{\geq}\ n\ r\ s]\!]^M &= [\![|\lambda_{x:e}\ r\ x \wedge s\ x| \geq n]\!]^M && \text{(e.g. } \textsf{Some}\ r\ s = \textsf{Count}_{\geq}\ 1\ r\ s\text{)}\\
[\![\textsf{Count}_{\leq}\ n\ r\ s]\!]^M &= [\![|\lambda_{x:e}\ r\ x \wedge\ s\ x| \leq n]\!]^M && \text{(e.g. } \textsf{None}\ r\ s = \textsf{Count}_{\leq}\ 0\ r\ s\text{)}\\
[\![\textsf{Ratio}_{\geq}\ n\ r\ s]\!]^M &= [\![|\lambda_{x:e} r\ x \wedge s\ x| \geq n \times |r|]\!]^M && \text{(e.g. } \textsf{All}\ r\ s = \textsf{Ratio}_{\geq}\ 1\ r\ s\text{)}\\
[\![\textsf{Ratio}_{\leq}\ n\ r\ s]\!]^M &= [\![|\lambda_{x:e} r x \wedge s\ x| \leq n \times |r|]\!]^M && \text{(e.g. } \textsf{Few}\ r\ s = \textsf{Ratio}_{\leq}\ .5\ r\ s\text{)}
\end{aligned}$$

with variants $q_{=}$, $q_{<}$, $q_{>}$ for other comparison operators defined in terms of these:

$$\begin{aligned}
[\![q_{=}\ n\ r\ s]\!]^M &= [\![q_{\leq}\ n\ r\ s \wedge q_{\geq}\ n\ r\ s]\!]^M\\
[\![q_{<}\ n\ r\ s]\!]^M &= [\![q_{\leq}\ n\ r\ s \wedge \neg q_{\geq}\ n\ r\ s]\!]^M\\
[\![q_{>}\ n\ r\ s]\!]^M &= [\![q_{\geq}\ n\ r\ s \wedge \neg q_{\leq}\ n\ r\ s]\!]^M
\end{aligned}$$

Now we can express arbitrary claims about probability:

$$\textsf{P}(\textsf{Edible} \mid \textsf{Nut}) > .20 \quad\Leftrightarrow\quad \textsf{Ratio}_{>}\ .20\ (\lambda_x\ \textsf{Nut}\ x)\ (\lambda_x\ \textsf{Edible}\ x)$$

Practice 5.5:

Classify the following as cardinal or proportional:

1. *one third*
2. *seven*

References

[Barwise & Cooper, 1981] Barwise, J. & Cooper, R. (1981). Generalized quantifiers and natural language. *Linguistics and Philosophy*, 4.

[Cantor, 1880] Cantor, G. (1880). Ueber unendliche, lineare Punktmannichfaltigkeiten – 2. (Fortsetzung des Artikels in Bd. XV, pag. 1.). *Mathematische Annalen*, 17(3), 355 – 358.