LING4400: Lecture Notes 8
First Order Logic

Contents

We use generalized quantifiers to model rich (including probabilistic) meanings.

Today we'll look at a weaker logic with only one quantifier, which is often used in math.

8.1 Basic (first-order) quantifiers [Peirce, 1870, Frege, 1879]

First-order quantifier functions make generalizations about all of the entities in the world model.

They map all possible characteristic functions (that is, sets) to truth values, so have type: $\langle\langle e, t\rangle, t\rangle$.

1. The **universal quantifier** returns true only for the $\langle e, t\rangle$ function that is true for all entities.

   Its table looks like this:

$[\![\text{Universal}]\!]^M =$

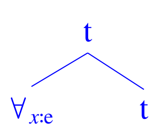
<table>
<tr><th>input</th><th></th><th>output</th></tr>
<tr><td>⋮</td><td></td><td>⋮</td></tr>
<tr><td>(all others)</td><td>:</td><td>False</td></tr>
<tr><td>⋮</td><td></td><td>⋮</td></tr>
<tr><td><table>
<tr><th>input</th><th></th><th>output</th></tr>
<tr><td>Africa</td><td>:</td><td>True</td></tr>
<tr><td>Asia</td><td>:</td><td>True</td></tr>
<tr><td>Laos</td><td>:</td><td>True</td></tr>
<tr><td>Mali</td><td>:</td><td>True</td></tr>
<tr><td>Togo</td><td>:</td><td>True</td></tr>
</table></td><td>:</td><td>True</td></tr>
</table>

   This is more commonly notated:

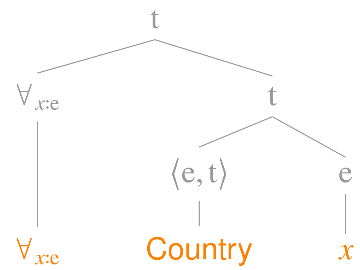
$$[\![\forall_{x:e}\ \varphi]\!]^M = [\![\text{Universal}\ (\lambda_{x:e}\ \varphi)]\!]^M$$

   (you will also sometimes see $\forall_{x:\alpha}\ \varphi$ notated with a dot after the variable: $\forall x{:}\alpha\ .\ \varphi$).

   We can draw this in a derivation tree using another kind of composition rule:



For example:



This means *Everything is a country.*

These are often used with implication to make claims that seem to be narrower.

For example, this expression means *All people are mortal*:

$$\forall_{x:e}\ \text{Person}\ x \to \text{Mortal}\ x$$

or, equivalently, using a generalized quantifier:

$$\text{All}\ (\lambda_{x:e}\ \text{Person}\ x)\ (\lambda_{x:e}\ \text{Mortal}\ x)$$

Here the implication is vacuously true for all values of x that are not people.

2. The **existential quantifier** returns false only for the $\langle e, t\rangle$ function that is false for all entities.

   Its table looks like this:

$[\![\text{Existential}]\!]^M =$

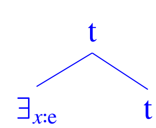
input	output
(input : output) Africa : False, Asia : False, Laos : False, Mali : False, Togo : False	: False
⋮	⋮
(all others)	: True
⋮	⋮

This is more commonly notated:

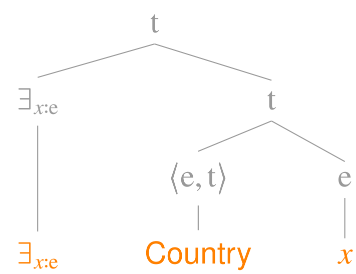
$$[\![\exists_{x:e}\ \varphi]\!]^M = [\![\text{Existential}\ (\lambda_{x:e}\ \varphi)]\!]^M$$

(you will also sometimes see $\exists_{x:\alpha}\ \varphi$ notated with a dot after the variable: $\exists x{:}\alpha\ .\ \varphi$).

We can draw this in a derivation tree using another kind of composition rule:



For example:



This means *Something is a country.*

These are often used with conjunction to resemble generalized quantifiers.

For example, this expression means *Some people are mortal*:

$$\exists_{x:\mathrm{e}}\ \mathsf{Person}\ x \wedge \mathsf{Mortal}\ x$$

or, equivalently, using a generalized quantifier:

$$\mathsf{Some}\ (\lambda_{x:\mathrm{e}}\ \mathsf{Person}\ x)\ (\lambda_{x:\mathrm{e}}\ \mathsf{Mortal}\ x)$$

It is derivable from the universal quantifier, like disjunction is derivable from conjunction:

$$[\![\exists_{x:\mathrm{e}}\ \varphi]\!]^M = [\![\neg\ \forall_{x:\mathrm{e}}\ \neg\ \varphi]\!]^M.$$

Logic with just these kinds of functions is called **first-order logic**.

Practice 8.1:

Assume a world model with two entities: (**A**, **B**), and two truth values.

Draw the truth table for the universal quantifier.

Practice 8.2:

Translate this expression from first-order logic into English: $\forall_{x:\mathrm{e}}\ \mathsf{City}\ x \rightarrow \mathsf{Capital}\ x$.

Practice 8.3:

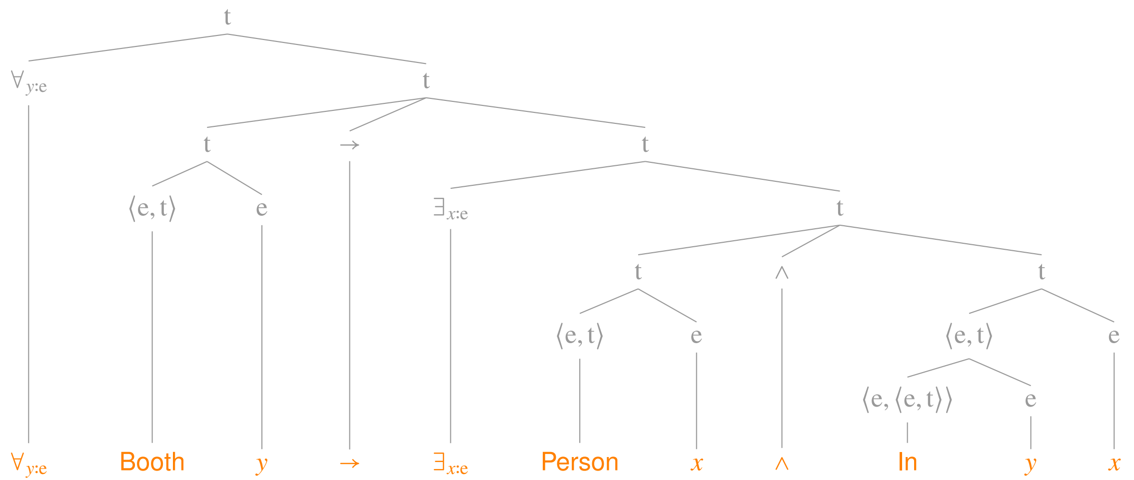
Write a logic expression using the propositional and first-order functions defined in the lecture notes, as well as constant Italy of type e and predicates Volcano of type ⟨e, t⟩ and Contain of type ⟨e, ⟨e, t⟩⟩ stating that *Italy contains a volcano*.

8.2 Building complex expressions out of first-order quantifiers

We can understand how logical expressions are constructed by drawing them as **derivation trees**:

1. The **branches** are function abstractions and applications and are labeled with types:
2. The **leaves** are constants, variables and logical symbols. (Later, we will use words.)

For example, here's a logical representation of '*Someone is in every booth.*':



These same derivation trees are used for calculating denotations during interpretation.

Practice 8.4: tree drawing

Draw a derivation tree for the following expression:

$$\forall_{x:e}\ \textsf{City}\ x \rightarrow \textsf{Capital}\ x$$

Practice 8.5: translating first-order quantifiers into generalized quantifiers

Translate the below first-order quantified expression:

$$\forall_{y:e}\ \textsf{Booth}\ y \rightarrow \exists_{x:e}\ \textsf{Person}\ x \wedge \textsf{In}\ y\ x$$

into an expression using only generalized quantifiers Some and All, and predicates Booth, Person and In.

8.3 Classes of relations

We can use first-order logic to describe several interesting classes of relations:

1. Classes related to reflexivity:
   (a) A relation r is **reflexive** if and only if $\forall_{x:e}\ r\ x\ x$.

   For example, = is reflexive: $3 = 3$.
   (b) A relation r is **nonreflexive** if and only if $\neg\ \forall_{x:e}\ r\ x\ x$.

   For example, Trusts is nonreflexive: ¬ Trusts Kim Kim.
   (c) A relation r is **irreflexive** if and only if $\forall_{x:e}\ \neg\ r\ x\ x$.

   For example, ≠ is irreflexive: $\neg\ 3 \neq 3$.
2. Classes related to symmetry:
   (a) A relation r is **symmetric** if and only if $\forall_{x:e}\ \forall_{y:e}\ r\ x\ y \to r\ y\ x$.

   For example, Borders is symmetric: Borders Togo Ghana → Borders Ghana Togo.
   (b) A relation r is **nonsymmetric** if and only if $\neg\ \forall_{x:e}\ \forall_{y:e}\ r\ x\ y \to r\ y\ x$.

   For example, Loves is nonsymmetric: ¬ (Loves Kim Pat → Loves Pat Kim).
   (c) A relation r is **asymmetric** if and only if $\forall_{x:e}\ \forall_{y:e}\ r\ x\ y \to \neg\ r\ y\ x$.

   For example, > is asymmetric: $3 > 2 \to \neg\ 2 > 3$.
3. Classes related to transitivity:
   (a) A relation r is **transitive** if and only if $\forall_{x:e}\ \forall_{y:e}\ \forall_{z:e}\ (r\ x\ y \land r\ y\ z) \to r\ x\ z$.

   For example, > is transitive: $3 > 2 \land 2 > 1 \to 3 > 1$.
   (b) A relation r is **nontransitive** if and only if $\neg\ \forall_{x:e}\ \forall_{y:e}\ \forall_{z:e}\ (r\ x\ y \land r\ y\ z) \to r\ x\ z$.

   For example, Borders is nontransitive:

   ¬ (Borders Ghana Togo ∧ Borders Togo Benin → Borders Ghana Benin).
   (c) A relation r is **intransitive** if and only if $\forall_{x:e}\ \forall_{y:e}\ \forall_{z:e}\ (r\ x\ y \land r\ y\ z) \to \neg\ r\ x\ z$.

   For example, Consecutive is intransitive: Consec 1 2 ∧ Consec 2 3 → ¬ Consec 1 3.

Practice 8.6:

Which of the above classes do the following relations belong to?

1. *overlaps*
2. *is next to*
3. *is larger than*

8.4 Formal properties of first-order logic

Mathematicians like first-order logic because it is **complete** [Gödel, 1929].

That means that every expression that is true in all world models can be derived from axioms.

It also has **semi-decidable consistency**: derivable expressions are true in all models [Gentzen, 1936].

(It does not have **fully decidable** consistency: inconsistent expressions may not be recognized.)

However, speakers don't generally use very sophisticated entailments that need these guarantees.

Mathematicians are ok with universal quantifiers because they study math facts that are always true.

Later we'll see an inability to compare set sizes leads to problems with linguistic expressivity.

References

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[Gödel, 1929] Gödel, K. (1929). *Über die Vollständigkeit des Logikkalküls*. PhD thesis, University Of Vienna.

[Peirce, 1870] Peirce, C. S. (1870). Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of boole's calculus of logic. *Memoirs of the American Academy of Arts and Sciences*, 9, 317–378.